human endeavours ("Die ganzen Zahlen hat der liebe Gott gemacht, alles andere ist Menschenwerk").[20] Today mathematicians would rather speak about natural numbers, namely (0), 1, 2, 3, ... [21] The emphasis on the key role of natural numbers and integers inspired Kronecker to explicitly introduce the ideal of arithmetisation:

> I also believe that eventually we will succeed in 'arithmetizing' the entire content of mathematical disciplines, that is to say, to provide a foundation for it purely and solely on the number concept, taken in the strictest sense (Cabillon 2011).[22]

Kronecker wrote something similar to Cantor from 'Kammer am Attersee' (21 August 1884), when he stated that his starting point is that everything in pure mathematics could be reduced to the theory of the integers and that he believes that it will be possible in all respects.[23]

7. At the crossroad: between discovery and construction

Kronecker is indeed a mathematician who contributed to what ultimately became known as constructivism in mathematics. Poincaré continued the Kantian legacy by distinguishing between logic and intuition. In his famous presentation on the infinite, David Hilbert reminds us that Kant already taught that mathematics disposes over a content which is independent of all logic and therefore can never be based solely on logic (Hilbert 1925: 171). He also points out:

20 In 2005 Stephen Hawking acted as editor of a work which partially used these words of Kronecker: *God created the integers, the mathematical breakthroughs that changed history* (Hawking 2005). A photocopy of the original 1887-article is in my possession.

21 In mathematics the number 0 plays an important role, *inter alia* as additive identity of the system of integers. If a set has the operation of addition, then 0 could be added to any element *x* of it without changing it.

22 "... ich glaube auch, dass es dereinst gelingen wird, den gesammten Inhalt aller dieser mathematischen Disciplinen zu 'arithmetisiren', d.h. einzig und allein auf den im engsten Sinne genommenen Zahlbegriff zu gründen, also die Modificationen und Erweiterungen dieses Begriffs wieder abzustreifen" (Kronecker 1887: 265). Cabillon J G 2011. http://mathforum.org/kb/message.jspa?messageID=1178416&tstart=0, quoting Kronecker on arithmetisation.

23 "Ich bin deshalb darauf ausgegangen, Alles in der reinen Mathematik auf die Lehre von den ganzen Zahlen zurükzuführen, und ich glaube, dass dies durchweg gelingen wird" (Meschkowski 1967: 238).

> Only when we analyze attentively do we realize that in presenting the laws of logic we already have had to employ certain arithmetical basic concepts, for example the concept of a set and partially also the concept of number, particularly as cardinal number [*Anzahl*]. Here we end up in a vicious circle and in order to avoid paradoxes it is necessary to come to a partially simultaneous development of the laws of logic and arithmetic (Hilbert 1970: 199, cf also Quine 1970: 88).

Speaking of 'the laws of logic and arithmetic' underscores the underlying problem running through our fore-going reflections: are these laws creations of human thought or are they rather discovered? In addition, are there perhaps other options?

The intuitive understanding of a law is that it delimits and determines whatever is subjected to it. To the constructivist it appears that the mathematician posits the stipulations or conditions of specific mathematical structures. These stipulations or conditions serve as the laws holding for what is correlated with them as subjects. Surely the conditions holding for something can never coincide with it. The conditions for being green are not themselves green, just as little as the physical laws for matter are themselves material. What is normally viewed as a mere fact – such as stating that 3+4=7 – actually relates certain numbers in a lawful way, conforming to the arithmetical law (operation) of addition. Although self-evident, it must be noted that the statement that 3+4=7 is a numerical (arithmetical) fact. However, in order to appreciate why this is not totally self-evident, we merely have to mention a similar sum – one obtained by first walking 3 miles north and then 4 miles east – in which case one would be 5 miles away from one's starting point. A vector is known to have both direction and distance. This explains at once that it is a spatial subject (a specific line-stretch) and not merely an arithmetical subject (like numbers).

Clearly, numerical subjects ought to be distinguished from spatial subjects, and this observation entails that we now have two different kinds of facts at hand: a numerical fact (designated as 3+4=7) and a geometrical fact (designated as 3 +4 = 5) (Figure 1).